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Notes on S-matrix of non-critical $N = 2$ string

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Abstract: In this paper we discuss the scattering S-matrix of non-critical $N = 2$ string at tree level. First we consider the $\hat{c} < 1$ string defined by combining the $N = 2$ time-like linear dilaton SCFT with the $N = 2$ Liouville theory. We compute three particle scattering amplitudes explicitly and find that they are actually vanishing. We also find an evidence that this is true for higher amplitudes. Next we analyze another $\hat{c} < 1$ string obtained from the $N = 2$ time-like Liouville theory, which is closely related to the $N = 2$ minimal string. In this case, we find a non-trivial expression for the three point functions. When we consider only chiral primaries, the amplitudes are very similar to those in the $(1, n)$ non-critical bosonic string.

Keywords: Topological Strings, Conformal Field Models in String Theory, 2D Gravit.

Contents

1. Introduction and summary

The $N = 2$ fermionic string [1] (for a review see [2]) is described by the most supersymmetric world-sheet among physically interesting string theories which have positive critical dimensions. Interestingly, its critical dimension is four because the ghost anomaly cancelation requires $\hat{c} = 2$. When we consider the ordinary flat spacetime, there is only one massless physical state and its spacetime theory is described by the self-dual gravity [3]. In this case, however, the $N = 2$ world-sheet supersymmetry requires two time directions.

In spite of this signature problem, we can still believe that $N = 2$ string theory provides the most beautiful and simple model at least from theoretical viewpoints. Indeed, the spectrum of $N = 2$ string in the four dimensional flat spacetime $R^{2,2}$ is very similar to those in two dimensional string theories in bosonic or type 0 string. These $N = 0, 1$ non-critical string theories can be non-perturbatively solvable by the matrix model descriptions [4 − 6] via the open-closed duality [7 − 10]. Therefore, it will be very intriguing if we can understand better a non-perturbative formulation of $N = 2$ string via an analogous open-closed duality.

Motivated by this, we would like to consider non-critical $N = 2$ string models. They are defined by the $N = 2$ Liouville theory plus a matter $N = 2$ SCFT with a central charge $\hat{c} < 1$, such that the total central charge is $\hat{c} = 2$ [11 − 15]. Indeed, our knowledge of such models is very little compared with that of $N = 0, 1$ string. There is no known matrix model like description though they are expected to be integrable. Recent discussions can be found in [16, 17].

In this paper we first consider the $N = 2$ $\hat{c} < 1$ string whose matter part is the time-like linear dilaton theory.[1] Physically, this model describes a three dimensional spacetime (one time, one space and one null direction) with a dynamical massless scalar field. Also it can be regarded as a $N = 2$ string analogue of the $c < 1$ deformation of the two dimensional string whose matrix model dual was constructed recently in [22]. Notice that we can regard the ordinary $N = 2$ string on $R^{2,2}$ as a particular $\hat{c} = 1$ model in this sense.

A basic dynamical property in these $N = 2$ string models will be the scattering amplitudes. We explicitly compute three particle scattering S-matrix at tree level in the equivalent $N = 4$ topological string formulation [23] , employing the recent progresses on three point functions in the $N = 2$ Liouville theory [24] or equivalent $N = 2$ $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ coset [25]. In the end, we find a very simple result that the three particle scattering amplitudes are all zero in this $N = 2$ $\hat{c} < 1$ string theories. There is an indirect evidence that four point functions are also zero. Even though we do not have any complete proof of this, it is natural to expect that this model is a free theory except the reflection or the two point function due to the Liouville wall. On the other hand, a similar $c < 1$ deformation via a time-like linear dilaton in bosonic string or type 0 string leads to non-trivial S-matrix [26, 22]. This suggests that the $N = 2$ non-critical string has a rather different structure than the $N = 0, 1$ string counterparts.

Furthermore, we also study another $N = 2$ string model whose matter part is given by the time-like $N = 2$ Liouville theory, which is defined by a Wick rotation of the ordinary space-like $N = 2$ Liouville theory as in the $N = 0$ case [27, 28]. This theory will be closely related to the minimal $\hat{c} < 1$ string (or $N = 2$ string on ALE space [15]), which are defined by a combination of a $N = 2$ minimal model and the $N = 2$ Liouville theory. In this case, we find that the three particle amplitudes are non-trivial in general.

The most ambitious goal will be to construct a non-perturbative description like matrix models.[2] It is natural to believe that this may be possible by applying an open-closed duality. Indeed, its simple spectrum, i.e. just a single scalar field, strongly suggests this. Since the rich spectrum of D-brane boundary states in $N = 2$ Liouville theory has been recently obtained [30 − 32, 24] , this may not be difficult (see e.g. [33, 34] for the D-branes in $N = 2$ string on $R^{2,2}$). We will leave this for a future problem.

[1] Linear dilaton backgrounds in Heterotic $N = 2$ string [18] may also have interesting properties as pointed out in [19] (see also [20, 21]).

[2] For a discussion on a holographic dual theory for $N = 2$ string on $R^{2,2}$ refer to [29].

This paper is organized as follows. In section 2 we give the definition of the $N = 2$ string whose matter part is the time-like linear dilaton SCFT. In section 3 we review $N = 4$ topological string formulation which is equivalent to $N = 2$ string and we explain how to apply it to our model. In section 4 we compute the scattering amplitudes in that theory. In section 5 we study another $N = 2$ string whose matter part is the $N = 2$ time-like Liouville theory.

2. $N = 2$ $\hat{c} < 1$ string with time-like linear dilaton matter

The $N = 2$ string has the critical central charge $c = 6$ (or $\hat{c} = 2$) and its simplest example is the four dimensional flat spacetime with two times and two spaces $R^{2,2}$. Even in this critical case, there is only one physical state (massless scalar field or self-dual graviton) and thus it looks similar to the $c = 1$ bosonic string or $\hat{c} = 1$ type 0 string. Here we consider the $\hat{c} < 1$ deformation by using the time-like linear dilaton SCFT as a matter part. In section 4 we will analyze another $N = 2$ $\hat{c} < 1$ string by adding a time-like Liouville term.

2.1 Definition

In general, $N = 2$ string is defined by specifying $N = 2$ SCFT which has the total central charge $\hat{c} = 2$ and by gauging the $N = 2$ Virasoro algebra generated by $(T, G^{\pm}, J)$. Here we assume the combination of the time-like linear dilaton $N = 2$ SCFT ($\hat{c} = 1 - Q^2$) and the $N = 2$ Liouville SCFT ($\hat{c} = 1 + Q^2$) as the $\hat{c} = 2$ $N = 2$ SCFT. We define the world-sheet fields in the time-like theory by (X^0, X^1) with fermions $(\psi_0, \bar{\psi}_0)$, and those in the space-like Liouville theory by (ϕ, Y) with fermions $(\psi, \bar{\psi})$. Their OPEs are

$$\begin{aligned} X^0(z)X^0(0) &\sim \log z\,, \qquad X^1(z)X^1(0) \sim \log z\,, \qquad \psi_0(z)\bar{\psi}_0(0) \sim -\frac{1}{z}\,, \\ \phi(z)\phi(0) &\sim -\log z\,, \qquad Y(z)Y(0) \sim -\log z\,, \qquad \psi(z)\bar{\psi}(0) \sim \frac{1}{z}\,. \end{aligned} \tag{2.1}$$

We can bosonize the fermions

$$\psi_0 = ie^{iH_0}\,, \qquad \bar{\psi}_0 = ie^{-iH_0}\,, \qquad \psi = e^{iH}\,, \qquad \bar{\psi} = e^{-iH}\,. \tag{2.2}$$

The string coupling constant g_s depends on both time and space due to the null linear dilaton in this background[3]

$$g_s = e^{\frac{Q}{2}(X^0 - \phi)}\,. \tag{2.3}$$

Then the $N = 2$ energy stress tensor and their super-partners are

$$\begin{aligned} T &= \frac{1}{2}(\partial X^0)^2 + \frac{Q}{2}\partial^2 X^0 + \frac{1}{2}(\partial X^1)^2 + \frac{1}{2}(\psi_0\partial\bar{\psi}_0 + \bar{\psi}_0\partial\psi_0) - \\ &\quad -\frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi - \frac{1}{2}(\partial Y)^2 - \frac{1}{2}(\psi\partial\bar{\psi} + \bar{\psi}\partial\psi)\,, \\ G^+ &= -i\psi_0\partial(X^0 - iX^1) - iQ\partial\psi_0 + i\psi\partial(\phi - iY) + iQ\partial\psi\,, \\ G^- &= -i\bar{\psi}_0\partial(X^0 + iX^1) - iQ\partial\bar{\psi}_0 + i\bar{\psi}\partial(\phi + iY) + iQ\partial\bar{\psi}\,, \\ J &= -\psi_0\bar{\psi}_0 - iQ\partial X^1 + \psi\bar{\psi} + iQ\partial Y\,. \end{aligned} \tag{2.4}$$

[3]In this paper we always set $\alpha' = 2$.

In the closed string theory there are two copies of the generators (2.4) as usual and they are denoted by $(T_L, G_L^{\pm}, J_L)$ and $(T_R, G_R^{\pm}, J_R)$.

Next we introduce the $N = 2$ Liouville potential to regulate the strongly coupled region[4]

$$\frac{\mu}{2\pi}\int dz^2 d\theta^2 e^{-\frac{\Phi}{Q}} + \text{h.c.}\,, \tag{2.5}$$

where Φ is the chiral super-field whose lowest component is given by $\phi + iY$. It is well-known that the $N = 2$ Liouville theory is T-dual to the $N = 2$ $\mathrm{SL}(2,\mathbb{R})_n/\mathrm{U}(1)$ coset [35]. The level n is related to the background charge via $Q = \sqrt{\frac{2}{n}}$. In our example we do not have to assume that n is always an integer because we do not couple the $N = 2$ Liouville theory to a $N = 2$ minimal model.

Finally we can define a $N = 2$ $\hat{c} < 1$ string by gauging the generators $(T_{L,R}, G_{L,R}^{\pm}, J_{L,R})$. In the original model of $N = 2$ Liouville theory the radius in Y direction is given by Q. We can also take a Z_m orbifold of the cigar model $\mathrm{SL}(2,\mathbb{R})_n/\mathrm{U}(1)$ in the circle direction. This is T-dual to the $N = 2$ Liouville theory with the radius mQ [35]. Taking the large m limit, we can decompactify the Y direction. In this paper we are mainly interested in the case where Y and X^1 are non-compact,[5] though we will also give results in the compactified case.

2.2 Physical vertex operators and physical spacetime dimension

Let us first consider the case where the radius of Y is infinite. The basic physical state in the $N = 2$ $\hat{c} < 1$ models is a single massless scalar field as in the ordinary $N = 2$ string on $R^{2,2}$. In the $N = 2$ string, since the U(1) R-current is gauged, we have only to consider the (NS, NS) sector. Other spin structures are equivalent.

The vertex operator in the $(-1,-1,-1,-1)$ picture can be written as

$$V = e^{-\phi_{1L}-\phi_{2L}-\phi_{1R}-\phi_{2R}}\cdot \exp\left[\left(\frac{Q}{2}+iE_0\right)X^0 + iE_1X^1 + \left(-\frac{Q}{2}+ik_2\right)\phi + ik_3Y\right], \tag{2.6}$$

where ϕ_1 and ϕ_2 are the bosonized superconformal ghosts in $N = 2$ string. The physical conditions that the conformal dimension Δ of V is one and its R-charge q is zero, lead to the constraints

$$E_0 = \pm k_2\,, \qquad E_1 = -k_3\,. \tag{2.7}$$

The second condition does not have any bosonic or $N = 1$ string counterpart, while the other one comes from the on-shell condition. This extra condition reduces the original 2+2

[4] In this model, the strongly coupled region at the time-like infinity is not regulated. However, if we consider the scattering of a massless particle off the Liouville wall, the particle will not get into the strongly coupled region.

[5] It is straightforward to see that the one-loop torus partition function in the non-compact case is the same as the ordinary case $Q = 0$ after integrating over the zero modes, assuming the continuous representation. Thus it is modular invariant in the $N = 2$ string sense.

dimensional spacetime to the three dimensional one which has one time, one space and one null direction.[6]

This operator (2.6) is the only physical state when we consider the continuous representation in the $N = 2$ Liouville theory as we can see from its partition function. There may be some discrete states, though we will not discuss them in this paper. If we compactified Y and X^1, then we need to consider winding modes also. The physical state conditions can be analyzed in the same way.

3. $N = 4$ topological string description

Since the $N = 2$ SCFT which appears in $N = 2$ string always has the central charge $\hat{c} = 2$, its symmetry enhances to the $N = 4$ superconformal algebra by adding the SU(2) currents $J^{++} = e^{\int J}$ and $J^{--} = e^{-\int J}$.

For example, in our model (2.4), the other generators are given by

$$\begin{aligned} J^{++} &= e^{iH_0} e^{iH} e^{iQY - iQX^1}\,, \\ J^{--} &= e^{-iH_0} e^{-iH} e^{-iQY + iQX^1}\,, \\ \tilde{G}^{+} &= e^{iQY - iQX^1} \left[e^{iH_0} \left(i(\partial\phi + i\partial Y) + Q\partial H \right) + e^{iH} \left(-(\partial X^0 + i\partial X^1) + iQ\partial H_0 \right) \right], \\ \tilde{G}^{-} &= e^{-iQY + iQX^1} \left[e^{-iH_0} \left(i(\partial\phi - i\partial Y) - Q\partial H \right) + e^{-iH} \left(-(\partial X^0 - i\partial X^1) - iQ\partial H_0 \right) \right]. \end{aligned} \tag{3.1}$$

By employing this $N = 4$ symmetry, we can define $N = 4$ topological string theory and this is known to be equivalent to $N = 2$ string [23] in general. Since in our example it turns out that the computations are easier and more systematic in the $N = 4$ topological string description, we will compute the scattering amplitudes in this formulation later.

3.1 $N = 4$ topological string and tree level scattering amplitudes

Here we review the definition of $N = 4$ topological string and the computations of on-shell scattering amplitudes. Its topological twist can be given by $T \to T + \frac{1}{2}\partial J$ as usual [38, 23]. After the twist the operators G^+ and $\tilde{G}^+$ have the conformal dimension $\Delta = 1$, while G^- and $\tilde{G}^-$ have $\Delta = 2$. Since the former ones satisfy $(G_0^+)^2 = (\tilde{G}_0^+)^2 = \{G_0^+, \tilde{G}_0^+\} = 0$, they behave like BRST operators. The physical state is a R-charge $+1$ state that satisfies

$$G_0^+ \Psi = \tilde{G}_0^+ \Psi = 0\,, \tag{3.2}$$

and the equivalence is defined by $\Psi \sim \Psi + G_0^+ \tilde{G}_0^+ \chi$.

[6] We can also consider another model by taking the T-duality in the Y direction. This is so called α type in [36] (see also [37]), while the ordinary one is called β type. In the α type case, the constraints are given by the asymmetric ones $E_0^{L,R} = \pm k_2^{L,R}$, $E_1^L = -k_3^L$, $E_1^R = k_3^R$. If we assume Y and X^1 are non-compact, we find that the physical spacetime is reduced to 1+1 dimensional. However, in this case it seems difficult to construct a modular invariant partition function. It is possible that the model is consistent when compactified. Since this model is related to the β type model via the T-duality, we find their scattering amplitudes directly from those in this paper. So we only talk about β type model in this paper.

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In this formalism we can define each of the M-particle genus g scattering amplitudes [23]. In particular, we are interested in the most basic one: three point function at tree level in closed string ($g = 0, M = 3$). It is defined by

$$A_3 = \langle |\widehat{\tilde{G}}_0^+|^2 V \cdot |\widehat{\tilde{G}}_0^+|^2 V \cdot V \rangle\,, \tag{3.3}$$

where the square means the left-moving and right-moving sector contributions. The $\Delta = q = 0$ operator[7] V is related to the physical state[8] Ψ in $N = 4$ topological string via

$$\Psi = |\widehat{\tilde{G}_0^+}|^2 V\,, \tag{3.4}$$

where we defined $\widehat{\tilde{G}^+}$ by

$$\widehat{\tilde{G}^+} = u_1 \tilde{G}^+ + u_2 G^+\,. \tag{3.5}$$

The twistor parameters (u_1, u_2) satisfy $|u_1|^2 + |u_2|^2 = 1$ and correspond to the SU(1,1) Lorentz transformation. The parameters for the left-moving sector or right-moving sector are denoted by (u_1^L, u_2^L) or (u_1^R, u_2^R).

It is easy to confirm that the three point function (3.3) in the $N = 4$ topological string indeed agrees with that in $N = 2$ string. The $N = 2$ string amplitudes with the zero instanton number is written as $\left\langle V \cdot V \cdot |[G^+_{-1/2}, G^-_{-1/2}]|^2 V \right\rangle$. Then we perform the topological twist and this procedure is equivalent to the spectral flow by $M - 2(1 - g) = 1$ unit, i.e. the insertion of $e^{\int J} = J^{++}$. After we move the positions of J^{++} and G^- using contour integral expressions and use the OPE $J^{++}(z)G^-(0) \sim z^{-1}\tilde{G}^+(0)$, we reproduce the terms in (3.3) which are proportional to $u_1^L u_2^L u_1^R u_2^R$. In the same way we can confirm that the other terms correspond to the amplitudes in non-zero instanton sectors.

Since (3.3) is the topologically twisted expression, we have to rewrite it in the untwisted NS-sector language in order to compute it in the CFT expressions. It is given by

$$A_3 = \langle |\widehat{\tilde{G}}^+_{-1/2} V \cdot \widehat{\tilde{G}}^+_{-1/2} V \cdot J^{--}_{-1} V|^2 \rangle\,, \tag{3.6}$$

after the spectral flow by $J^{--} = e^{-\int J}$. Then it is obvious that the total R-charge is zero consistently.

3.2 An example of $N = 4$ topological string amplitudes: flat space $R^{2,2}$

Let us first consider the $N = 2$ string in flat space[9] $R^{2,2}$ $(Z^1, \bar{Z}^1, Z^2, \bar{Z}^2)$ from the viewpoint of $N = 4$ topological string and compute the amplitudes. In this case we have

$$V = e^{i(k_1 \bar{Z}^1 + \bar{k}_1 Z^1 + k_2 \bar{Z}^2 + \bar{k}_2 Z^2)}\,. \tag{3.7}$$

[7] This operator V is interpreted as the $(-1, -1, -1, -1)$ picture vertex operator in the $N = 2$ string as will be clear from the later arguments.

[8] In many examples we can find a corresponding V operator when Ψ is given [23]. Indeed it is easy to check that the vertex (2.6) satisfies $G_0^+ \tilde{G}_0^+ V = \tilde{G}_0^+ G_0^+ V = 0$ by using the on-shell conditions in our examples.

[9] Metric is $g_{1\bar{1}} = 1,\ g_{2\bar{2}} = -1$. Here we do not assume any linear dilaton.

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The superconformal generators are

$$
\begin{aligned}
G^+ &= \psi^1 \partial \bar{Z}^1 - \psi^2 \partial \bar{Z}^2 \,, \qquad & G^- &= \bar{\psi}^1 \partial Z^1 - \bar{\psi}^2 \partial Z^2 \,, \\
\tilde{G}^+ &= \psi^1 \partial Z^2 - \psi^2 \partial Z^1 \,, \qquad & \tilde{G}^- &= \bar{\psi}^1 \partial \bar{Z}^2 - \bar{\psi}^2 \partial \bar{Z}^1 \,.
\end{aligned} \tag{3.8}
$$

Notice also that $J^- = \psi^1 \psi^2$.

Then we obtain

$$
\Psi = \left| u_1(k_1\psi_2 + k_2\psi_1) + u_2(\bar{k_1}\psi^1 + \bar{k_2}\psi^2) \right|^2 e^{i(k\bar{Z} + \bar{k}Z)} \,. \tag{3.9}
$$

The three point function can be computed as

$$
A_3 = \langle \Psi^{(1)} \Psi^{(2)} | J^-_{-1} |^2 V^{(3)} \rangle = c_{12}(u_1^L, u_2^L) \cdot c_{12}(u_1^R, u_2^R) \,, \tag{3.10}
$$

where we have defined

$$
\begin{aligned}
c_{12}(u_1, u_2) = {} & (u_1)^2 [k_2^{(1)} k_1^{(2)} - k_1^{(1)} k_2^{(2)}] + (u_2)^2 [-\bar{k}_2^{(1)} \bar{k}_1^{(2)} + \bar{k}_1^{(1)} \bar{k}_2^{(2)}] + \\
& + u_1 u_2 [\bar{k}_1^{(1)} k_1^{(2)} - \bar{k}_1^{(2)} k_1^{(1)} - (\bar{k}_2^{(1)} k_2^{(2)} - \bar{k}_2^{(2)} k_2^{(1)})] \,.
\end{aligned} \tag{3.11}
$$

The momenta $k^a_{1,2}$ represent those of the a-th ($a = 1, 2, 3$) particle.

Indeed the term proportional to $u_1^L u_1^R u_2^L u_2^R$ in (3.10) corresponds to the well-known $N = 2$ string amplitude with the zero instanton number [3], while the other terms proportional to $(u_1^L)^{1+n_L} (u_2^L)^{1-n_L} (u_1^L)^{1+n_R} (u_2^L)^{1-n_R}$ correspond to the amplitudes in the instanton number (n_L, n_R) sector induced by the U(1) gauge flux on the $N = 2$ Riemann surface.

4. Scattering amplitudes

The scattering S-matrix is the most basic quantity which characterizes string theory models. In this section we discuss the tree level scattering amplitudes in the $N = 2$ $\hat{c} < 1$ string defined in the previous section. The S-matrix for $M_1 + M_2$ particle scattering describes the amplitude when we send M_1 particles from the large ϕ region and observe that M_2 particles are reflected back from the Liouville wall[10] as in the two dimensional string.

4.1 Classification of amplitudes

As we have shown in section 2.2, the on-shell vertex operators (2.6) can be classified into two types due to the on-shell conditions (2.7): $k_2 = E_0$ and $k_2 = -E_0$. We call each of them (+) type (in-coming) or (−) type (out-going), respectively. When we evaluate the amplitudes in terms of the correlation functions in the Liouville theory, it is useful to employ the quantum numbers $(j, m, \bar{m})$ in the equivalent $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model. When compactified in Y and X^1 direction, the relation between them is given by

$$
ik_2 = iE_0 = Q\left(j + \frac{1}{2}\right) , \qquad k_3^L = -E_1^L = Qm \,, \qquad k_3^R = -E_1^R = Q\bar{m} \,, \tag{4.1}
$$

[10] Here we implicitly neglect the physical interpretation of the extra null direction. It looks harmless compared with the extra time-direction in the ordinary $N = 2$ string on $R^{2,2}$.

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for a (+) particle, while it is

$$ik_2 = -iE_0 = Q\left(j + \frac{1}{2}\right), \qquad k_3^L = -E_1^L = Qm, \qquad k_3^R = -E_1^R = Q\bar{m}, \tag{4.2}$$

for a (−) particle.[11] By using these quantum numbers instead of the energy and momenta, we can express the vertex operator for a in-coming (or out-going) particle by $V^{(+)}_{j,m,\bar{m}}$ (or $V^{(-)}_{j,m,\bar{m}}$). It is also convenient to define[12] the complex valued momenta[13]

$$k = ik_2 + k_3 - \frac{Q}{2}, \qquad \bar{k} = ik_2 - k_3 - \frac{Q}{2}, \qquad E = iE_0 + E_1 + \frac{Q}{2}, \qquad \bar{E} = iE_0 - E_1 + \frac{Q}{2}. \tag{4.3}$$

4.2 Reflection relation

As in the bosonic or $N = 1$ case, the physical excitations are reflected back off the Liouville wall in the $N = 2$ Liouville theory. This leads to the following equivalence relation between

$$V^{(+)}_{j,m,\bar{m}} \sim R(j, m, \bar{m}) V^{(-)}_{-j-1,m,\bar{m}}, \tag{4.4}$$

where we defined the reflection coefficient (or two point function) [25, 30, 24]

$$R(j, m, \bar{m}) = -\mu^{\frac{2(2j+1)}{n}} \frac{\Gamma(1 + j + m)\Gamma(1 + j - \bar{m})}{\Gamma(-j + m)\Gamma(-j - \bar{m})} \frac{\Gamma\left(-\frac{2j+1}{n}\right)\Gamma(-2j - 1)}{\Gamma\left(\frac{2j+1}{n}\right)\Gamma(2j + 1)}. \tag{4.5}$$

4.3 Three particle scattering amplitudes

The three point functions are classified into the following four types

$$(+ + +), \quad (+ + -), \quad (- - +), \quad (- - -). \tag{4.6}$$

For each type the X^0 momentum conservation can be written as

$$j_1 + j_2 + j_3 = -2, \quad j_1 + j_2 - j_3 = -1, \quad j_1 + j_2 - j_3 = 0, \quad j_1 + j_2 + j_3 = -1. \tag{4.7}$$

Since the (+) and (−) are related to each other by the reflection $j \to -j - 1$ as we have seen in the previous subsection, we just have to compute in one of the four case. We will argue that all of the three particle amplitudes are actually vanishing.

Let us first give a short and intuitive argument why we find the vanishing amplitudes. When we consider the $(- - -)$ amplitude, the energy conservation condition i.e. the last one in (4.7): $j_1 + j_2 + j_3 = -1$ coincides with the momentum conservation in the space-like linear dilaton theory. Then we do not need any insertion of the Liouville potential to compute it. Thus we expect that the $(- - -)$ amplitude is the same as that calculated

[11] On the other hand, in the α string case, we just have to replace k_3^R with $-k_3^R$ in (4.1) and (4.2).

[12] For example, we get $(k^L, \bar{k}^L, E^L, \bar{E}^L) = Q(j + m, j - m, j - m + 1, j + m + 1)$ for (+) type and for the (−) type $(k^L, \bar{k}^L, E^L, \bar{E}^L) = Q(j + m, j - m, -j - m, -j + m)$, for the (−) type.

[13] On-shell conditions are given by $k\bar{k} - E\bar{E} + Qk + QE = k\bar{k} - E\bar{E} + Q\bar{k} + Q\bar{E} = 0$.

in the linear dilaton theory. The three point function in linear dilaton $N = 2$ string is generally proportional to $(c_{12})^2$, where we defined

$$c_{ab} = E^{(a)}\bar{E}^{(b)} - E^{(b)}\bar{E}^{(a)} - k^{(a)}\bar{k}^{(b)} + k^{(b)}\bar{k}^{(a)} . \tag{4.8}$$

For on-shell particles they satisfy

$$c_{ab} = -c_{ba} , \qquad \sum_{a=1}^{4} c_{ab} = 0 . \tag{4.9}$$

The momentum $k^{(a)}$ and energy $E^{(a)}$ $(a = 1, 2, 3)$ are those for a-th particle in the complex valued notation (4.3). Since $k^{(a)}$ and $\bar{k}^{(a)}$ are proportional to $E^{(a)}$ and $\bar{E}^{(a)}$ for the $(-)$ particle, c_{ab} is vanishing. Therefore the $(- - -)$ amplitude is vanishing.[14] The others will also be zero because of the reflection relation. Notice that we have non-vanishing three particle scattering amplitudes in $N = 0$ (bosonic) and $N = 1$ (type0) string. This shows that the non-critical $N = 2$ string has a rather different property than the $N = 0, 1$ one.

More rigorous proof of the vanishing amplitudes including those in non-zero instanton sectors can be done by the explicit computation of the amplitudes using the $N = 2$ Liouville theory. We will show the details in the appendix A.2 and here we gave a brief sketch of the calculations. We employ the equivalent description of the $N = 4$ topological string explained in the previous section.

To make the presentation simple, we assume the X^1 direction is non-compact. Then we only allow the momentum modes in X^1 and Y direction. The results for the compact case including winding modes can be found in a very similar way and they are also vanishing.

In this set up, we can write the amplitude (3.6) in the following form

$$A_3 = (u_1^L)^2(u_1^R)^2 A^{11} + (u_2^L)^2(u_2^R)^2 A^{22} + u_1^L u_2^L u_1^R u_2^R A^{12} . \tag{4.10}$$

For example, the A^{11} can be expressed by the kinematical factors and the three point functions in $N = 2$ Liouville theory.

$$\begin{aligned} A^{11} &\equiv \langle |\tilde{G}^+_{-1/2} V \cdot \tilde{G}^+_{-1/2} V \cdot J^-_{-1} V|^2 \rangle \\ &= \delta\left(\sum_{a=1}^{3} m_a + 1\right) \cdot \delta\left(\sum_{a=1}^{3} \bar{m}_a + 1\right) \cdot \delta\left(iE_0^{(a)} + \frac{Q}{2}\right) \cdot [-a + b + c - d] , \end{aligned} \tag{4.11}$$

where a, b, c and d are defined by

$$\begin{aligned} a &= (\bar{k}^{(1)})^2(\bar{E}^{(2)})^2 \cdot \langle V^{(0,0)}_{j_1,m_1+1,m_1+1} V^{(1,1)}_{j_2,m_2,m_2} V^{(-1,-1)}_{j_3,m_3,m_3} \rangle , \\ b &= \bar{k}^{(1)}\bar{k}^{(2)}\bar{E}^{(1)}\bar{E}^{(2)} \cdot \langle V^{(0,1)}_{j_1,m_1+1,m_1} V^{(1,0)}_{j_2,m_2,m_2+1} V^{(-1,-1)}_{j_3,m_3,m_3} \rangle , \\ c &= \bar{k}^{(1)}\bar{k}^{(2)}\bar{E}^{(1)}\bar{E}^{(2)} \cdot \langle V^{(1,0)}_{j_1,m_1,m_1+1} V^{(0,1)}_{j_2,m_2+1,m_2} V^{(-1,-1)}_{j_3,m_3,m_3} \rangle , \\ d &= (\bar{k}^{(2)})^2(\bar{E}^{(1)})^2 \cdot \langle V^{(1,1)}_{j_1,m_1,m_1} V^{(0,0)}_{j_2,m_2+1,m_2+1} V^{(-1,-1)}_{j_3,m_3,m_3} \rangle , \end{aligned} \tag{4.12}$$

[14] If we do the same computation for other cases in the totally linear dilaton model, the amplitude $(+ + -)$ turns out to be non-zero in a specific case [14]. However, this is not consistent with the reflection relation in the Liouville theory. We believe at that point the computation becomes singular and the correct answer is zero.

where $V^{(s,\bar{s})}_{j,m,\bar{m}}$ denotes the primary operator in the $N=2$ Liouville theory defined by

$$V^{(s,\bar{s})}_{j,m,\bar{m}} = e^{Qj\phi+iQ(m+s)Y_L+iQ(\bar{m}+\bar{s})Y_R+isH_L+i\bar{s}H_R}\,. \tag{4.13}$$

The three point functions in the $N=2$ Liouville theory have already computed in [25, 24] as reviewed in appendix A.1. After we substitute them into (4.12), we can find that A^{11} is actually zero for all four cases (4.6). In the $(---)$ case this is very easy to see because all of the Liouville three point functions in (4.12) are the same. It is indeed proportional to the $(c_{12})^2$ as expected from the previous linear dilaton calculation. In the same way we can confirm that A^{12} and A^{22} are vanishing as we have shown in the appendix A.2.

4.4 Discussions on four particle scattering

Even though the explicit four point function in $N=2$ Liouville theory is not known, we can discuss the four particle scattering from the analysis of the linear dilaton theory. Motivated by this, let us compute the on-shell four particle scattering in the linear dilaton theory i.e. $\mu=0$. We define the Mandelstam valuables ($a=1,2,3,4$ represents the label of the four particles)

$$\begin{aligned} s &= -(E^{(1)}\bar{E}^{(3)}+E^{(3)}\bar{E}^{(1)})+k^{(1)}\bar{k}^{(3)}+k^{(3)}\bar{k}^{(1)}\,,\\ t &= -(E^{(2)}\bar{E}^{(3)}+E^{(3)}\bar{E}^{(2)})+k^{(2)}\bar{k}^{(3)}+k^{(3)}\bar{k}^{(2)}\,,\\ u &= -(E^{(4)}\bar{E}^{(3)}+E^{(3)}\bar{E}^{(4)})+k^{(4)}\bar{k}^{(3)}+k^{(3)}\bar{k}^{(4)}\,. \end{aligned} \tag{4.14}$$

They satisfy the relations $s+t+u=0$ for on-shell particles.

Then the S-matrix of the four particles is given by

$$A_4 = F^2\cdot\frac{\Gamma(1-s/2)\Gamma(1-t/2)\Gamma(1-u/2)}{\Gamma(s/2)\Gamma(t/2)\Gamma(u/2)}\,, \tag{4.15}$$

where F is defined by

$$F = 1-\frac{c_{24}c_{13}}{su}-\frac{c_{23}c_{14}}{tu}\,. \tag{4.16}$$

This resulting form is the same as the well-known result[15] in flat space [3], though the values of momenta E,k are different from that one. We expect that the $(----)$ amplitude in the $N=2$ Liouville theory should coincide with the one in the linear dilaton theory as we confirmed in the previous example. In this case, c_{ab} and (s,t,u) are all vanishing. By treating them as infinitesimals of the same order, we can find that the amplitude is zero. Then all other four particle amplitudes will be zero due to the reflection relation. This may be analogous to the fact that all $M\geq 4$ particle scattering amplitudes in flat space $R^{2,2}$ are zero at any genus [23]. This result strongly suggests that the $N=2$ $\hat{c}<1$ string with the time-like linear dilaton is actually a free theory[16] except the reflection (or two point function) at the Liouville wall. It would be interesting to study this further.

We would also like to note that the naive $Q=0$ limit of our model is not equivalent to the familiar $N=2$ string on $R^{2,2}$. The latter has a different dimension and non-zero three particle scattering amplitudes.

[15] Notice that we need the exchange $1\leftrightarrow 4$ due to the different convention.

[16] This is true for both α and β type model. In the α string model this may not be so surprising because we know that this theory is free on $R^{2,2}$ without linear dilaton as shown in [37, 39].

5. Scattering amplitudes in $\hat{c} < 1$ string with time-like Liouville matter

We can define another $N = 2$ $\hat{c} < 1$ string by combining a time-like $N = 2$ Liouville theory with the standard (space-like) $N = 2$ Liouville Theory. This time-like $N = 2$ Liouville theory can be defined by the Wick rotation of the space-like one as was done for the bosonic string case [27, 28, 40]. This $\hat{c} < 1$ string is closely related to the $N = 2$ minimal $\hat{c} < 1$ string after the Z_n orbifold. The latter is defined by combining the $N = 2$ minimal models and the $N = 2$ Liouville theory, which is equivalent to the $N = 2$ string on ALE spaces [15 – 17]. Since the n-th $N = 2$ minimal models or equally $N = 2$ $\mathrm{SU}(2)/\mathrm{U}(1)$ coset at level n, is also regarded as the negative level $n \to -n$ continuation of $N = 2$ $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset or $N = 2$ Liouville theory. Thus we expect the correlation functions in our model are essentially the same as those in the minimal $N = 2$ string at tree level, though the number of physical states is restricted to be finite only in the minimal model case.

5.1 Time-like $N = 2$ Liouville theory

We can define the time-like $N = 2$ Liouville theory (X^0, X^1) with the background charge Q_0 via the Wick-rotation of the ordinary $N = 2$ Liouville theory (Y, ϕ) with the background charge Q_2

$$X_0 = -i\phi\,, \quad X_1 = -iY\,, \quad H_0 = H\,, \quad Q_2 = iQ_0\,. \tag{5.1}$$

In this case, the momenta are rotated as follows

$$E_0 = ik_2 = iQ_0\left(j + \frac{1}{2}\right), \qquad E_1^L = k_3^L = -Q_0 m\,, \qquad E_1^R = k_3^R = -Q_0\bar{m} \tag{5.2}$$

where $(j, m, \bar{m})$ are the usual quantum numbers of the $\mathrm{SL}(2,\mathbb{R})_{n+2}$ WZW model ($Q_2 = \sqrt{2/n}$). In the actual computations of correlation functions, we set $Q_0 = Q$ and regard the time-like vertex operators as a $\mathrm{SL}(2,\mathbb{R})_{2-n}$ primary via the rule

$$iE_0 = -Q(j + \frac{1}{2})\,, \qquad E_1 = -Qm\,. \tag{5.3}$$

Notice the flip of the sign $n \to -n$. The quantum number s is the same as the space-like theory.

The $(+)$ type ($E_0 = k_2$) physical vertex in this theory is given by

$$V_j^{(+)} = \tilde{V}_{-j-1,m,\bar{m}} \otimes V_{j,m,\bar{m}}\,, \tag{5.4}$$

and the $(-)$ type ($E_0 = -k_2$) one is

$$V_j^{(-)} = \tilde{V}_{j,m,\bar{m}} \otimes V_{j,m,\bar{m}}\,. \tag{5.5}$$

Here $\tilde{V}$ denotes the primary in the time-like Liouville theory, which can be treated as the ordinary $N = 2$ Liouville theory with the imaginary background charge iQ. V is the primary in the ordinary $N = 2$ Liouville theory with the background charge Q. If we are interested in the $N = 2$ string on ALE space (or $\frac{\mathrm{SU}(2)_{n-2}}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2,\mathbb{R})_{n+2}}{\mathrm{U}(1)}$), the values of (j, m) become half-integers and follow constraints e.g. $0 \le j \le n/2$ as usual.

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5.2 Three particle scattering amplitudes

Now we can compute the three particle scattering amplitudes in the $N = 4$ topological string formulation. We present the detailed calculations in the appendix A.3. Here we just write down the final result. We assume the compactification of X^1 and Y is arbitrary.[17] The result of the $(---)$ amplitude can be written as

$$A_{3(---)} = A_{\Delta s=0} + A_{\Delta s=\pm 1} \,. \tag{5.6}$$

The first term comes from the fermion number conserving part of the three point function (i.e. the first term in (A.1)) and the second one from the fermion number violating part[18] (the second and third term in (A.1)).

The latter is explicitly given by

$$\begin{aligned} A_{\Delta s=\pm 1} = Q^4 (u_1^L - i u_2^L)^2 (u_1^R - i u_2^R)^2 \cdot \left[\delta^2 \left(\sum_a m_a + 1 - \frac{n}{2} \right) + \delta^2 \left(\sum_a m_a + 1 + \frac{n}{2} \right) \right] \cdot \\ \cdot \tilde{D}_+(j_a) \cdot D_+(j_a) \cdot \prod_{a=1}^{3} \frac{\Gamma(j_a - m_a + 1)\Gamma(j_a + m_a + 1)}{\Gamma(-j_a + \bar{m}_a)\Gamma(-j_a - \bar{m}_a)} \,, \end{aligned} \tag{5.7}$$

where $\tilde{D}_+(j_a)$ is the wick-rotated (or time-like) version[19] of the function $D_+(j_a)$. The function $D_+(j_a)$ was first defined in [24] and is also reviewed in the appendix A.1 of the present paper. Notice that in this case we have left-right asymmetric terms in addition to the previous expression (4.10) . To get amplitudes for the (+) particle, we just have to replace $\frac{\Gamma(j_a - m_a + 1)}{\Gamma(-j_a + \bar{m}_a)}$ with $\frac{\Gamma(-j_a - m_a)}{\Gamma(j_a + \bar{m}_a + 1)}$, and $\tilde{D}_+(j_a)$ with $\tilde{D}_+(-j_a - 1)$ in (5.7).

The former term $A_{\Delta s=0}$ in general can be written in terms of the functions $D(j_a)$ and $F(j_a, m_a, \bar{m}_a)$ as we computed in the appendix A.3. Actually, it is vanishing[20] in the following important cases: (i) $j_a = m_a = \bar{m}_a$ for one of $a = 1, 2, 3$, (ii) either of the four conditions in (4.7) is satisfied.

Finally, let us evaluate (5.7) in the special case of $j_a = m_a = \bar{m}_a$. This correspond to the chiral primary states.[21] In this case we can obtain the explicit formula of $\tilde{D}_+(j_a)$ and $D_+(j_a)$.

The amplitudes are vanishing except (i) $\sum_a j_a + 1 - \frac{n}{2} = 0$ or (ii) $\sum_a j_a + 1 + \frac{n}{2} = 0$ is satisfied. In these two cases we obtain[22] $A^{(i)}_{3(---)} = \alpha^{(-)} \cdot \mu$ and $A^{(ii)}_{3(---)} = -\alpha^{(-)} \cdot \mu^{-1}$,

[17] Here we present results in the β type model again, but those in α type model can be obtained easily via T-duality.

[18] Notice that in the time-like linear dilaton case we only had the fermion number conserving part since the fermion number in the time-like CFT is conserved.

[19] This Wick-rotation procedure should be careful and non-trivial as in [28, 40].

[20] In more general, we can show $A_{\Delta s=0} = 0$ when $F(j_a, m_a, \bar{m}_a)$ is factorized as $F(j_a, m_a, \bar{m}_a) = f(j_a, m_a) g(j_a, \bar{m}_a)$ for some functions f and g.

[21] This computation looks very similar to that of three point functions in the $N = 2$ topologically twisted $\mathrm{SL}(2, \mathbb{R})/\mathrm{U}(1)$ model in [41]. Indeed, we can derive the same result in [41] using a similar analysis in $N = 2$ Liouville theory. Notice that its result is somewhat different from the $N = 4$ case as their definitions are not the same.

[22] Naively we may just regard these amplitudes are vanishing. This is consistent with its original definition

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where $\alpha^{(-)}$ represents a double zero $\sim \Gamma(0)^{-2}$ due to the Gamma-function in (5.7). On the other hand, another one $A_{3(+++)}$ with the opposite chiralities is given by in each cases

$$\begin{aligned} A^{(i)}_{3(+++)} &= \alpha^{(+)} \cdot \mu \cdot \prod_{a=1}^{3} \gamma\left(1 - \frac{2j_a+1}{n}\right), \\ A^{(ii)}_{3(+++)} &= \alpha^{(+)} \cdot \mu^{-1} \cdot \prod_{a=1}^{3} \gamma\left(1 - \frac{2j_a+1}{n}\right), \end{aligned} \tag{5.8}$$

where $\alpha^{(+)}$ is a divergent constant ($\sim \Upsilon(0)^{-1}$). We also defined $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$.

It will be exciting to note that the amplitude $A^{(ii)}_{3(+++)}$ with the momentum conservation $\sum_a j_a + 1 + \frac{n}{2} = 0$ in (5.8) agrees with those in the $(1,n)$ non-critical bosonic string. This suggests that the chiral primary sector in the minimal $N=2$ string is essentially equivalent to the $(1,n)$ string. A similar observation can be obtained from the ground-ring structure computed recently in [17]. Indeed it includes a Z_{n-1} subring, which is the same as the ground ring in $(1,n)$ string. Also it is known that the $(1,n)$ string is described by so called ADE matrix model [42]. The ADE series of this model seem to match with the ADE classification of ALE spaces. It would be interesting to explore this relation further.

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A. Details of computations of three particle amplitudes

A.1 Three point functions in $N=2$ Liouville theory

The three point functions in $N=2$ Liouville theory are given by[23] [24] (see also [25, 43])

$$\begin{aligned} &\langle V^{(s_1,\bar{s}_1)}_{j_1,m_1,\bar{m}_1} V^{(s_2,\bar{s}_2)}_{j_2,m_2,\bar{m}_2} V^{(s_3,\bar{s}_3)}_{j_3,m_3,\bar{m}_3} \rangle = \\ &\quad = \delta^2\left(\sum_{a=1}^{3} m_a\right) \cdot \delta^2\left(\sum_{a=1}^{3} s_a\right) \cdot D(j_a) \cdot F(j_a, m_a, \bar{m}_a) + \\ &\qquad + \delta^2\left(\sum_{a=1}^{3} m_a - 1 - \frac{n}{2}\right) \cdot \delta^2\left(\sum_{a=1}^{3} s_a + 1\right) \cdot D_-(j_a) \cdot F_-(j_a, m_a, \bar{m}_a) + \\ &\qquad + \delta^2\left(\sum_{a=1}^{3} m_a + 1 + \frac{n}{2}\right) \cdot \delta^2\left(\sum_{a=1}^{3} s_a - 1\right) \cdot D_+(j_a) \cdot F_+(j_a, m_a, \bar{m}_a). \end{aligned} \tag{A.1}$$

in the $N=4$ topological string because the $\hat{\tilde{G}}_+$ annihilates the chiral primary operator $j=m$. However, as we will see shortly the $A_{3(+++)}$ is not vanishing. and this suggests us that $A_{3(---)}$ should not be regarded as just a zero.

[23] Here we omit the world-sheet coordinate dependence and the cocycle factors. It turns out that they are irrelevant for our computations.

The functions $D(j_a)$, $D_\pm(j_a)$ and $F_\pm(j_a,m_a,\bar m_a)$ are given by

$$D(j_a) = \frac{(\nu b^{-2b^2})^{j_1+j_2+j_3+1}\Upsilon(0)'\Upsilon(b(2j_1+1))\Upsilon(b(2j_2+1))\Upsilon(b(2j_3+1))}{\sqrt{2}b^2\Upsilon(b^{-1}-b(j_{1+2+3}+1))\Upsilon(b^{-1}+bj_{1-2-3})\Upsilon(b^{-1}+bj_{2-1+3})\Upsilon(b^{-1}+bj_{3-1-2})},$$
$$D_\pm(j_a) = \frac{(\nu b^{2-2b^2})^{j_1+j_2+j_3+1}\Upsilon(0)'\Upsilon(b(2j_1+1))\Upsilon(b(2j_2+1))\Upsilon(b(2j_3+1))}{\sqrt{2}b^{1+n}\Upsilon(\frac{1}{2b}-b(j_{1+2+3}+1))\Upsilon(\frac{1}{2b}+bj_{1-2-3})\Upsilon(\frac{1}{2b}+bj_{2-1+3})\Upsilon(\frac{1}{2b}+bj_{3-1-2})},$$
$$F_\pm(j_a,m_a,\bar m_a) = (-1)^{m_2-\bar m_2}\prod_{a=1}^{3}\frac{\Gamma(1+j_a\pm m_a)}{\Gamma(-j_a\mp \bar m_1)}, \tag{A.2}$$

where we have defined $b=1/\sqrt{n}$ and $\nu=\mu^{2/n}$; we also used the notation like $j_{3-1-2}\equiv j_3-j_1-j_2$. The function $\Upsilon(x)$ is familiar one already in the bosonic Liouville theory [44]. It satisfies the relations $\Upsilon(x+b)=\gamma(bx)b^{1-2bx}\Upsilon(x)$ and $\Upsilon(x+1/b)=\gamma(x/b)b^{2x/b-1}\Upsilon(x)$ ($\gamma(x)\equiv\Gamma(x)/\Gamma(1-x)$). Another function $F(j_a,m_a,\bar m_a)$ is defined by the following integral

$$F(j_a,m_a,\bar m_a) = \pi^{-2}\int dz^2dw^2 z^{j_1-m_1}\bar z^{j_1-\bar m_1}(1-z)^{j_2-m_2}(1-\bar z)^{j_2-\bar m_2}\cdot$$
$$\cdot w^{j_1+m_1}\bar w^{j_1+\bar m_1}(1-w)^{j_2+m_2}(1-\bar w)^{j_2+\bar m_2}|z-w|^{-4-2(j_1+j_2+j_3)}. \tag{A.3}$$

Explicit evaluations of $D(j_a)$ and $F(j_a,m_a,\bar m_a)$. To perform actual calculations it is important to rewrite $D(j_a)$ and $F(j_a,m_a,\bar m_a)$ in terms of more basic functions. This is possible when the energy conservation (i.e. (4.7)) of the time-like linear dilaton theory is satisfied. The function $D(j_a)$ for each case is given by

$$j_1+j_2+j_3=-2:\ D(j_a)=2^{-1/2}\nu^{-1}n^{-3}\cdot\frac{\Upsilon'(0)}{\Upsilon(Q)}\cdot\prod_{a=1}^{3}[\gamma(-2j_a)\cdot\gamma(-(2j_a+1)/n)],$$
$$j_1+j_2-j_3=-1:\ D(j_a)=2^{-1/2}\nu^{2j_3}n^2\cdot\frac{\Upsilon'(0)}{\Upsilon(Q)}\cdot\gamma\left(1-\frac{2j_1+1}{n}\right)\cdot\gamma\left(1-\frac{2j_2+1}{n}\right),$$
$$j_1+j_2-j_3=0:\ D(j_a)=2^{-1/2}\nu^{2j_3+1}n^{3/2}\cdot\frac{\Upsilon'(0)}{\Upsilon(b^{-1})}\cdot\gamma\left(1-\frac{2j_3+1}{n}\right),$$
$$j_1+j_2+j_3=-1:\ D(j_a)=2^{-1/2}n^{1/2}\cdot\frac{\Upsilon'(0)}{\Upsilon(b^{-1})}\cdot\prod_{a=1}^{3}\gamma(-2j_a). \tag{A.4}$$

Notice the first two are divergent because $\Upsilon(Q)=0$. Notice also $\Upsilon'(0)=\Upsilon(b^{-1})=\Upsilon(b)$.

The function $F(j_a,m_a,\bar m_a)$ can also be explicitly evaluated by employing the formula

$$\int dx^2|x|^{2a}x^n|1-x|^{2b}(1-x)^m=\pi\cdot\frac{\Gamma(a+n+1)\Gamma(b+m+1)\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)\Gamma(a+b+m+n+2)}, \tag{A.5}$$

and the reflection relation. The results are given by

$$j_1+j_2+j_3=-2:\ F(j_a,m_a,\bar m_a)=\prod_{a=1}^{3}\frac{\Gamma(j_a-m_a+1)\Gamma(j_a+m_a+1)}{\Gamma(-j_a-\bar m_a)\Gamma(-j_a+\bar m_a)},$$
$$j_1+j_2-j_3=-1:\ F(j_a,m_a,\bar m_a)=(-1)^{m_3-\bar m_3}\cdot\gamma(-2j_1-1)\cdot\gamma(-2j_2-1)\cdot$$
$$\cdot\prod_{a=1}^{2}\frac{\Gamma(j_a-m_a+1)\Gamma(j_a+m_a+1)}{\Gamma(-j_a-\bar m_a)\Gamma(-j_a+\bar m_a)},$$

$$j_1+j_2-j_3=0: \quad F(j_a,m_a,\bar m_a) = (-1)^{m_1-\bar m_1}\cdot\Gamma(0)\cdot\gamma(-2j_1-1)\cdot \cdot\frac{\Gamma(j_1-m_1+1)\Gamma(j_1+m_1+1)}{\Gamma(-j_1-\bar m_1)\Gamma(-j_1+\bar m_1)},$$

$$j_1+j_2+j_3=-1: \quad F(j_a,m_a,\bar m_a) = \Gamma(0)\cdot\prod_{a=1}^{3}\gamma(2j_a+1)\,. \tag{A.6}$$

It is also useful to compute this factor in $j_2=m_2=\bar m_2$ case. The result is given by

$$F(j_a,m_a,\bar m_a)|_{j_2=m_2=\bar m_2} = (-1)^{m_3-\bar m_3}\frac{\Gamma(j_1-m_1+1)\Gamma(j_3-m_3+1)}{\Gamma(-j_1+\bar m_1)\Gamma(-j_3+\bar m_3)}\cdot\gamma(2j_2+1)\cdot \cdot\gamma(j_1-j_2-j_3)\cdot\gamma(-j_1-j_2-j_3-1)\cdot\gamma(-j_1-j_2+j_3)\,. \tag{A.7}$$

A.2 Three particle scatterings in $N=2$ $\hat c<1$ string with time-like linear dilaton

Now we would like to evaluate the three particle scattering amplitudes (4.10) by applying the previous formula in $N=2$ Liouville theory. To compute the amplitude we need the following expressions

$$\begin{aligned}
G^+_{-1/2}V &= \left(-ike^{iH}+Ee^{iH_0}\right)\cdot e^{(\frac{Q}{2}+iE_0)X^0+iE_1X^1+(-\frac{Q}{2}+ik_2)\phi+ik_3Y}\,,\\
\tilde G^+_{-1/2}V &= \left(-i\bar k e^{iH_0}-\bar E e^{iH}\right)\cdot e^{(\frac{Q}{2}+iE_0)X^0+i(E_1-Q)X^1+(-\frac{Q}{2}+ik_2)\phi+i(k_3+Q)Y}\,,\\
J^-_{-1}V &= e^{-iH_0}e^{-iH}e^{(\frac{Q}{2}+iE_0)X^0+i(E_1+Q)X^1+(-\frac{Q}{2}+ik_2)\phi+i(k_3-Q)Y}\,.
\end{aligned}\tag{A.8}$$

Then it is straightforward to see that the A^{11} is given by (4.11) and (4.12). Below we also present other amplitudes A^{22} and A^{12} explicitly. The amplitude A^{22} can be computed as follows (again we show results in β string, but we can find the results in α string easily via T-duality)

$$\begin{aligned}
A^{22} &\equiv \langle|G^+_{-1/2}V\cdot G^+_{-1/2}V\cdot J^-_{-1}V|^2\rangle \qquad\qquad (A.9)\\
&= \delta\Big(\sum_{a=1}^{3}m_a-1\Big)\cdot\delta\Big(\sum_{a=1}^{3}\bar m_a-1\Big)\cdot\delta\Big(\sum_{a=1}^{3}iE_0^{(a)}+\frac{Q}{2}\Big)\cdot[-a'+b'+c'-d']\,,
\end{aligned}$$

where a',b',c',d' are given by

$$\begin{aligned}
a' &= (E^{(1)})^2(k^{(2)})^2\cdot\langle V^{(0,0)}_{j_1,m_1,m_1}V^{(1,1)}_{j_2,m_2-1,m_2-1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle\,,\\
b' &= E^{(1)}E^{(2)}k^{(1)}k^{(2)}\cdot\langle V^{(0,1)}_{j_1,m_1,m_1-1}V^{(1,0)}_{j_2,m_2-1,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle\\
c' &= E^{(1)}E^{(2)}k^{(1)}k^{(2)}\cdot\langle V^{(1,0)}_{j_1,m_1-1,m_1}V^{(0,1)}_{j_2,m_2,m_2-1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle\\
d' &= (E^{(2)})^2(k^{(1)})^2\cdot\langle V^{(1,1)}_{j_1,m_1-1,m_1-1}V^{(0,0)}_{j_2,m_2,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle\,.
\end{aligned}\tag{A.10}$$

Also A^{12} is given by the more complicated expression[24]

$$A^{12}\equiv\langle|(G^+_{-1/2}V\cdot\tilde G^+_{-1/2}V\cdot J^-_{-1}V+\tilde G^+_{-1/2}V\cdot G^+_{-1/2}V\cdot J^-_{-1}V)|^2\rangle$$

[24] Notice that if all the correlators in the space-like Liouville theory are the same, the result is proportional to $(c_{12})^2$.

$$
\begin{aligned}
&= \delta\Big(\sum_{a=1}^{3} m_a\Big)\cdot\delta\Big(\sum_{a=1}^{3}\bar{m}_a\Big)\cdot\delta\Big(\sum_{a=1}^{3} iE_0^{(a)}+\frac{Q}{2}\Big)\cdot\\
&\cdot\Big[\Big(k^{(1)}\bar{k}^{(2)}\Big)^2\cdot\langle V^{(1,1)}_{j_1,m_1-1,m_1-1}V^{(0,0)}_{j_2,m_2+1,m_2+1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - k^{(1)}\bar{k}^{(2)}E^{(1)}\bar{E}^{(2)}\cdot\langle V^{(1,0)}_{j_1,m_1-1,m_1}V^{(0,1)}_{j_2,m_2+1,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - k^{(1)}\bar{k}^{(2)}E^{(1)}\bar{E}^{(2)}\cdot\langle V^{(0,1)}_{j_1,m_1,m_1-1}V^{(1,0)}_{j_2,m_2+1,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle +\\
&\quad + \Big(E^{(1)}\bar{E}^{(2)}\Big)^2\cdot\langle V^{(0,0)}_{j_1,m_1,m_1}V^{(1,1)}_{j_2,m_2,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle +\\
&\quad + \Big(k^{(2)}\bar{k}^{(1)}\Big)^2\cdot\langle V^{(0,0)}_{j_1,m_1+1,m_1+1}V^{(1,1)}_{j_2,m_2-1,m_2-1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - k^{(2)}\bar{k}^{(1)}E^{(2)}\bar{E}^{(1)}\cdot\langle V^{(0,1)}_{j_1,m_1+1,m_1}V^{(1,0)}_{j_2,m_2-1,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - k^{(2)}\bar{k}^{(1)}E^{(2)}\bar{E}^{(1)}\cdot\langle V^{(1,0)}_{j_1,m_1,m_1+1}V^{(0,1)}_{j_2,m_2,m_2-1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle +\\
&\quad + \Big(E^{(2)}\bar{E}^{(1)}\Big)^2\cdot\langle V^{(1,1)}_{j_1,m_1,m_1}V^{(0,0)}_{j_2,m_2,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle +\\
&\quad + k^{(1)}\bar{k}^{(2)}\bar{E}^{(1)}E^{(2)}\cdot\langle V^{(1,1)}_{j_1,m_1-1,m_1}V^{(0,0)}_{j_2,m_2+1,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - k^{(1)}\bar{k}^{(1)}k^{(2)}\bar{k}^{(2)}\cdot\langle V^{(1,0)}_{j_1,m_1-1,m_1+1}V^{(0,1)}_{j_2,m_2+1,m_2-1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - E^{(1)}\bar{E}^{(1)}\bar{E}^{(2)}E^{(2)}\cdot\langle V^{(0,1)}_{j_1,m_1,m_1}V^{(1,0)}_{j_2,m_2,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle +\\
&\quad + E^{(1)}\bar{E}^{(2)}k^{(2)}\bar{k}^{(1)}\cdot\langle V^{(0,0)}_{j_1,m_1,m_1+1}V^{(1,1)}_{j_2,m_2,m_2-1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle +\\
&\quad + k^{(1)}\bar{k}^{(2)}\bar{E}^{(1)}E^{(2)}\cdot\langle V^{(1,1)}_{j_1,m_1,m_1-1}V^{(0,0)}_{j_2,m_2,m_2+1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - k^{(1)}\bar{k}^{(1)}k^{(2)}\bar{k}^{(2)}\cdot\langle V^{(0,1)}_{j_1,m_1+1,m_1-1}V^{(1,0)}_{j_2,m_2-1,m_2+1}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle -\\
&\quad - E^{(1)}\bar{E}^{(1)}\bar{E}^{(2)}E^{(2)}\cdot\langle V^{(1,0)}_{j_1,m_1,m_1}V^{(0,1)}_{j_2,m_2,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle +\\
&\quad + E^{(1)}\bar{E}^{(2)}k^{(2)}\bar{k}^{(1)}\cdot\langle V^{(0,0)}_{j_1,m_1+1,m_1}V^{(1,1)}_{j_2,m_2-1,m_2}V^{(-1,-1)}_{j_3,m_3,m_3}\rangle\Big].
\end{aligned}
\tag{A.11}
$$

Evaluation of the amplitudes. Consider the $(---)$ case. Then as we can see from (A.4) and (A.6), the three point function in the Liouville theory does not depend on $m_a(=\bar{m}_a)$. Thus we can factorize the kinematical factors and find that they are all vanishing because $(k^{(a)},\bar{k}^{(a)})=-(E^{(a)},\bar{E}^{(a)})$ for (-) particles. Furthermore, application of the reflection relation or the direct evaluations show that all other types of amplitudes are vanishing.

In all four types of on-shell amplitudes the three point function $\sim D(j_a)F(j_a,m_a,\bar{m}_a)$ in the Liouville theory includes a common divergence. This is because[25] we do not need any insertion of Liouville operator in that computation when $j_1+j_2+j_3=-1$.

[25] One may wonder a careful treatment of this divergence in (A.4) and in (A.6) may can be canceled by the zeros in the kinematical factors leading to non-vanishing finite answers. However, this does not seem to be the case. Let us regulate by assuming the extra $\hat{c}=D$ directions. The linear dilaton gradient is now given by $g_s=e^{\frac{q}{2}X_0-\frac{Q}{2}\phi}$. Define $q=(1-\epsilon)Q$ and treat ϵ as an infinitesimal constant. Then it is easy to see that the correlation function for $(---)$ is given by just replacing $\Gamma(0)$ in (A.6) with $(\epsilon)^{-1}\cdot f(j_a,m_a)$, where f is a non-singular function. Then the kinematical factor is proportional to ϵ^2, while the Liouville correlator to ϵ^{-1}. Thus the total amplitude is of order ϵ and is vanishing.

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In the above computations we assumed that Y and X^1 are non-compact directions. In the compact case we have to take winding modes into account as in (4.2). But this can be done almost in the same way as before and in the end the three particle amplitudes are all vanishing.

A.3 Three particle scatterings in $N = 2$ $\hat{c} < 1$ string with time-like Liouville

Finally we present the detailed derivation of the result (5.7) in section 5. In this time-like Liouville case, one important point is that we can allow the violation of the fermion numbers s_a and $\bar{s}_a$. Indeed as we can see from the explicit computation below, only such new contributions become non-zero.

To make the expressions simple, we show results only for the A^{11} amplitude in the β type model, which is proportional to $(u_1^L)^2(u_1^R)^2$. The other part of the amplitude can be computed in the same way. We take the compactification in Y and X^1 direction arbitrary, including the non-compact case. It can be expressed in terms of correlation functions for the space-like and time-like Liouville theory as follows

$$\begin{aligned} A^{11} &\equiv \langle |\tilde{G}^+_{-1/2} V \cdot \tilde{G}^+_{-1/2} V \cdot J^-_{-1} V|^2 \rangle \\ &= \delta^2 \left(\sum_{a=1}^{3} m_a + 1 \right) \cdot [-a + b + c - d] + \\ &\quad + \delta^2 \left(\sum_{a=1}^{3} m_a + 1 - \frac{n}{2} \right) \cdot [e] + \delta^2 \left(\sum_{a=1}^{3} m_a + 1 + \frac{n}{2} \right) \cdot [f] \,, \end{aligned} \tag{A.12}$$

where a, b, c, d is defined by

$$\begin{aligned} a &= \bar{k}_L^{(1)} \bar{E}_L^{(2)} \bar{k}_R^{(1)} \bar{E}_R^{(2)} \cdot \langle \tilde{V}^{(1,1)}_{\tilde{j}_1, m_1, \bar{m}_1} \tilde{V}^{(0,0)}_{\tilde{j}_2, m_2+1, \bar{m}_2+1} \tilde{V}^{(-1,-1)}_{\tilde{j}_3, m_3, \bar{m}_3} \rangle \cdot \\ &\quad \cdot \langle V^{(0,0)}_{j_1, m_1+1, \bar{m}_1+1} V^{(1,1)}_{j_2, m_2, \bar{m}_2} V^{(-1,-1)}_{j_3, m_3, \bar{m}_3} \rangle \,, \\ b &= \bar{k}_L^{(1)} \bar{E}_L^{(2)} \bar{k}_R^{(2)} \bar{E}_R^{(1)} \cdot \langle \tilde{V}^{(1,0)}_{\tilde{j}_1, m_1, \bar{m}_1+1} \tilde{V}^{(0,1)}_{\tilde{j}_2, m_2+1, \bar{m}_2} \tilde{V}^{(-1,-1)}_{\tilde{j}_3, m_3, \bar{m}_3} \rangle \cdot \\ &\quad \cdot \langle V^{(0,1)}_{j_1, m_1+1, \bar{m}_1} V^{(1,0)}_{j_2, m_2, \bar{m}_2+1} V^{(-1,-1)}_{j_3, m_3, \bar{m}_3} \rangle \,, \\ c &= \bar{k}_L^{(2)} \bar{E}_L^{(1)} \bar{k}_R^{(1)} \bar{E}_R^{(2)} \cdot \langle \tilde{V}^{(0,1)}_{\tilde{j}_1, m_1+1, \bar{m}_1} \tilde{V}^{(1,0)}_{\tilde{j}_2, m_2, \bar{m}_2+1} \tilde{V}^{(-1,-1)}_{\tilde{j}_3, m_3, \bar{m}_3} \rangle \cdot \\ &\quad \cdot \langle V^{(1,0)}_{j_1, m_1, \bar{m}_1+1} V^{(0,1)}_{j_2, m_2+1, \bar{m}_2} V^{(-1,-1)}_{j_3, m_3, \bar{m}_3} \,, \\ d &= \bar{k}_L^{(2)} \bar{E}_L^{(1)} \bar{k}_R^{(2)} \bar{E}_R^{(1)} \cdot \langle \tilde{V}^{(0,0)}_{\tilde{j}_1, m_1+1, \bar{m}_1+1} \tilde{V}^{(1,1)}_{\tilde{j}_2, m_2, \bar{m}_2} \tilde{V}^{(-1,-1)}_{\tilde{j}_3, m_3, \bar{m}_3} \rangle \cdot \\ &\quad \cdot \langle V^{(1,1)}_{j_1, m_1, \bar{m}_1} V^{(0,0)}_{j_2, m_2+1, \bar{m}_2+1} V^{(-1,-1)}_{j_3, m_3, \bar{m}_3} \rangle \,, \end{aligned} \tag{A.13}$$

and the e and f are given by

$$\begin{aligned} e &= \bar{k}_L^{(1)} \bar{k}_L^{(2)} \bar{k}_R^{(1)} \bar{k}_R^{(2)} \cdot \langle \tilde{V}^{(1,1)}_{\tilde{j}_1, m_1, \bar{m}_1} \tilde{V}^{(1,1)}_{\tilde{j}_2, m_2, \bar{m}_2} \tilde{V}^{(-1,-1)}_{\tilde{j}_3, m_3, \bar{m}_3} \rangle \cdot \\ &\quad \cdot \langle V^{(0,0)}_{j_1, m_1+1, \bar{m}_1+1} V^{(0,0)}_{j_2, m_2+1, \bar{m}_2+1} V^{(-1,-1)}_{j_3, m_3, \bar{m}_3} \rangle \,, \\ f &= \bar{E}_L^{(1)} \bar{E}_L^{(2)} \bar{E}_R^{(1)} \bar{E}_R^{(2)} \cdot \langle \tilde{V}^{(0,0)}_{\tilde{j}_1, m_1+1, \bar{m}_1+1} \tilde{V}^{(0,0)}_{\tilde{j}_2, m_2+1, \bar{m}_2+1} \tilde{V}^{(-1,-1)}_{\tilde{j}_3, m_3, \bar{m}_3} \rangle \cdot \\ &\quad \cdot \langle V^{(1,1)}_{j_1, m_1, \bar{m}_1} V^{(1,1)}_{j_2, m_2, \bar{m}_2} V^{(-1,-1)}_{j_3, m_3, \bar{m}_3} \rangle \,. \end{aligned} \tag{A.14}$$

In the above equations $k_{L,R}$ means the left and right-moving part of the momentum. The value $\tilde{j}$ is equal to $-j-1$ for $(+)$ type and to j for $(-)$ type.

Computations of e and f. It is easy to calculate e and f. The results in the $(---)$ case are

$$e = f = Q^4 \tilde{D}_{\pm}(j_a) \cdot D_{\mp}(j_a) \cdot \prod_{a=1}^{3} \frac{\Gamma(j_a + m_a + 1)\Gamma(j_a - m_a + 1)}{\Gamma(-j_a + \bar{m}_a)\Gamma(-j_a - \bar{m}_a)}, \tag{A.15}$$

where $\tilde{D}_{\pm}(j_a)$ is the Wick-rotated version (i.e. $b \to ib$) of the $D_{\pm}$. On the other hand, in the $(+++)$ case we obtain

$$e = f = Q^4 \tilde{D}_{\pm}(-j_a - 1) \cdot D_{\mp}(j_a). \tag{A.16}$$

It is also straightforward to obtain the result for other cases. In the end we obtain (5.7).

Computations of a, b, c and d. Consider the $(---)$ amplitude again, because we can reproduce other ones by the reflection relation (4.4). For generic values of $(j_a, m_a, \bar{m}_a)$ we find

$$\begin{aligned} a = d &= Q^4 (j-m)^2 (j-\bar{m})^2 \cdot \tilde{D}(j_a) D(j_a) \cdot \\ &\quad \cdot F(j_a, m_1, \bar{m}_1, m_2+1, \bar{m}_2+1, m_3, \bar{m}_3) F(j_a, m_1+1, \bar{m}_1+1, m_2, \bar{m}_2, m_3, \bar{m}_3), \\ b = c &= Q^4 (j-m)^2 (j-\bar{m})^2 \cdot \tilde{D}(j_a) D(j_a) \cdot \\ &\quad \cdot F(j_a, m_1+1, \bar{m}_1, m_2, \bar{m}_2+1, m_3, \bar{m}_3) F(j_a, m_1, \bar{m}_1+1, m_2+1, \bar{m}_2, m_3, \bar{m}_3), \end{aligned} \tag{A.17}$$

where $\tilde{D}(j_a)$ is again the Wick-rotated version of $D(j_a)$. It seems that this cannot be rewritten by simple functions in generic cases; it is known that the function $F(j_a, m_a, \bar{m}_a)$ is expressed in terms of the hypergeometric function shown in [45].

However, in the particular case where the function F is factorized as $F(j_a, m_a, \bar{m}_a) = f(j_a, m_a) g(j_a, \bar{m}_a)$, it is clear that $a = b = c = d$ and thus these contributions from a, b, c and d in (A.12) are vanishing. Such examples can be found e.g. when (i) $j_a = m_a = \bar{m}_a$ for one of $a = 1, 2, 3$ and when (ii) either of the four conditions in (4.7) is satisfied.

The results for other instanton sectors we can obtain the results in the same way by just shifting m_a and $\bar{m}_a$ by ± 1 and flip some of the signs appropriately. So we omit writing down the full expression.

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